Pergamon
www.elsevier.com/locate/jappmathmech

J. Appl. Maths Mechs, Vol. 63, No. 6, pp. 937–942, 1999

0021–8928/99/\$—see front matter
PII: S0021–8928(00)00011-3

THE ASYMPTOTIC SOLUTION OF THE CONTACT PROBLEM FOR A THREE-DIMENSIONAL ELASTIC BODY OF FINITE DIMENSIONS†

I. I. ARGATOV

St Petersburg

(*Received* 1 *December* 1998)

The asymptotic form of Green's vector function with a pole on the boundary is calculated by the method of matched asymptotic expansions. The expansion obtained is used to construct the asymptotic form of the contact pressure. The equations of the contact problem are derived with integral corrections, which take into account the nature of the attachment and the geometry of the elastic body. Examples of calculations for an elliptic punch are given. © 2000 Elsevier Science Ltd. All rights reserved.

Asymptotic solutions of the contact problem were constructed previously (using the "large λ" method [1]) for semi-infinite elastic bodies, such as a layer (see [2], etc.) and a wedge [3]. The axisymmetrical problem of the impression of a punch into the end of a semi-infinite cylinder was investigated in [4].

1. FORMULATION OF THE PROBLEM

Consider an elastic body Ω, the boundary of which contains a plane section Σ situated in the Ox_1x_2 plane (see the figure). We will assume, for simplicity, that the body Ω belongs entirely to the half-space $x_3 > 0$ and we will let l be the radius of the greatest half-sphere within Ω with centre at the point O. Suppose ω is a plane figure contained in a circle of radius l. We will assume that $\varepsilon > 0$ is a small parameter and we will denote by $\omega(\varepsilon)$ the figure obtained from ω by contracting it ε^{-1} times. The vector $\mathbf{u} = (u_1, u_2, u_3)$ of the displacements of points of the body Ω due to the action of a punch with a base $\omega(\varepsilon)$, pressing without friction into the plane section Σ, satisfies the equations

$$\mu\nabla_x \cdot \nabla_x \mathbf{u}(\varepsilon; \mathbf{x}) + (\lambda+\mu)\nabla_x\nabla_x \cdot \mathbf{u}(\varepsilon; \mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \tag{1.1}$$

$$\sigma_{31}(\mathbf{u}; \mathbf{x}) = \sigma_{32}(\mathbf{u}; \mathbf{x}) = 0, \quad \mathbf{x} \in \Sigma \tag{1.2}$$

$$\sigma_{33}(\mathbf{u}; \mathbf{x}) = 0, \quad \mathbf{x} \in \Sigma, \quad \mathbf{x}' = (x_1, x_2) \notin \omega(\varepsilon) \tag{1.3}$$

$$u_3(\varepsilon; \mathbf{x}', 0) = \delta_0 + \beta_1 x_2 - \beta_2 x_1, \quad \mathbf{x}' \in \omega(\varepsilon) \tag{1.4}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u}; \mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_\sigma; \quad \mathbf{u}(\varepsilon; \mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_u \tag{1.5}$$

Here λ and μ are the Lamé parameters, $\sigma_{3k}(u)$ are the stresses, $\boldsymbol{\sigma}^{(n)}$ is the stress vector on an area with normal $\mathbf{n}$, δ_0 is the vertical displacement of the centre of the punch base, and β_1 and β_2 are the angles of rotation of the punch about the coordinate axes Ox_1 and Ox_2. We will assume that the body is attached to the section Γ_u, while on Γ_σ and Σ outside the contact area it is stress-free.

2. THE INTEGRAL EQUATION FOR DETERMINING THE CONTACT PRESSURE

We will denote Green's vector function with pole at the point $P(y_1, y_2, 0)$ by $\mathbf{G}(\mathbf{y}; \mathbf{x})$. This satisfies Eq. (1.1), conditions (1.5) and the following relations

$$\sigma_{3k}(\mathbf{G}, \mathbf{x}) = 0 \quad (k = 1, 2, 3), \quad \mathbf{x} \in \Sigma \setminus P \tag{2.1}$$

†*Prikl. Mat. Mekh.* Vol. 63, No. 6, pp. 1001–1007, 1999.

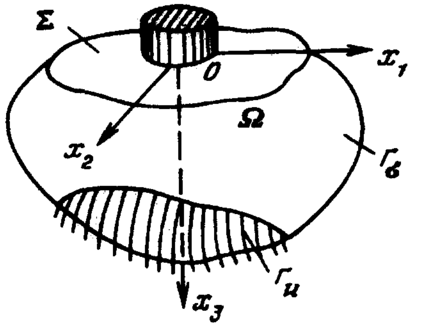

Fig. 1.

$$\mathbf{G}(\mathbf{y};\mathbf{x}) = \mathbf{T}(x_1 - y_1, x_2 - y_2, x_3) + O(1), \quad \mathbf{x} \to P \tag{2.2}$$

Here **T** is the solution of the Boussinesq problem (see, for example, [5]) on the loading of an elastic half-space with a unit point force, oriented along the Ox_3 axis, where

$$\begin{aligned} &4\pi\mu T_i(\mathbf{x}) = x_i x_3 |\mathbf{x}|^{-3} - (1-2\nu)x_i |\mathbf{x}|^{-1}(|\mathbf{x}|+x_3)^{-1}, \quad i=1,2 \\ &4\pi\mu T_3(\mathbf{x}) = x_3^2 |\mathbf{x}|^{-3} + 2(1-\nu)|\mathbf{x}|^{-1}; \quad |\mathbf{x}| = (x_1^2+x_2^2+x_3^2)^{1/2} \end{aligned} \tag{2.3}$$

We know (see, for example, [6, 7]), that using Green's vector function the problem of determining the contact pressure can be reduced to the integral equation

$$\iint\limits_{\omega(\varepsilon)} G_3(\mathbf{y}; x_1, x_2, 0)p(\mathbf{y})d\mathbf{y} = \delta_0 + \beta_1 x_2 - \beta_2 x_1 \tag{2.4}$$

Using the representation

$$\mathbf{G}(\mathbf{y};\mathbf{x}) = \mathbf{T}(x_1 - y_1, x_2 - y_2, x_3) + \mathbf{g}(\mathbf{y};\mathbf{x}) \tag{2.5}$$

where $\mathbf{g}(\mathbf{y};\mathbf{x})$ is a regular vector function, Eq. (2.4) becomes

$$\begin{aligned} &\frac{1-\nu^2}{\pi E} \iint\limits_{\omega(\varepsilon)} \frac{p(y_1, y_2)dy_1 dy_2}{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}} = \\ &= \delta_0 + \beta_1 x_2 - \beta_2 x_1 - \iint\limits_{\omega(\varepsilon)} g_3(\mathbf{y}; x_1, x_2, 0)p(\mathbf{y})d\mathbf{y} \end{aligned} \tag{2.6}$$

Here E is Young's modulus and ν is Poisson's ratio. For details of the properties of the solutions of Eq. (2.6) see [6, Section 51] and [8, Section 1.2].

3. THE ASYMPTOTIC FORM OF GREEN'S VECTOR FUNCTION

Suppose $\mathbf{g}(\mathbf{x})$ is the regular component of Green's vector function $\mathbf{G}(O;\mathbf{x})$ with a pole at the origin of coordinates, which annuls the residue $\mathbf{T}(\mathbf{x})$ in boundary conditions (1.5) and (1.6), i.e.

$$\boldsymbol{\sigma}^{(n)}(\mathbf{g};\mathbf{x}) = -\boldsymbol{\sigma}^{(n)}(\mathbf{T};\mathbf{x}), \quad \mathbf{x} \in \Gamma_\sigma; \quad \mathbf{g}(\mathbf{x}) = -\mathbf{T}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_u$$

Since **g** satisfies homogeneous equation (1.1) and boundary condition (2.1) in the neighbourhood of the point O, its Maclaurin series

$$\mathbf{g}(\mathbf{x}) = \mathbf{g}(0) + \sum_{k=1}^{6} g_{1,k}\mathbf{V}^{1,k}(\mathbf{x}) + \sum_{m=2}^{\infty}\sum_{k=1}^{3(m+1)} g_{m,k}\mathbf{V}^{m,k}(\mathbf{x}) \tag{3.1}$$

generally speaking, will contain $3(m+1)$ homogeneous vector polynomials $\mathbf{V}^{m,k}(\mathbf{x})$ of degree m (see [9, Chapter 13] and [10, Section 5.3]). At the boundary the normal component of the vector $\mathbf{g}(\mathbf{x})$ leaves the trace

$$\frac{2\pi\mu}{1-\nu}g_3(x_1,x_2,0)=A_0+B_1x_1+B_2x_2+C_{11}x_1^2+2C_{12}x_1x_2+C_{22}x_2^2+\dots \tag{3.2}$$

Remark. For $m = 1$ and $m = 2$ the vector function $\mathbf{V}^{m,k}$ can be chosen as follows:

$$\mathbf{V}^{1,1}=-x_3\mathbf{e}_2+x_2\mathbf{e}_3,\quad \mathbf{V}^{1,2}=x_3\mathbf{e}_1-x_1\mathbf{e}_3,\quad \mathbf{V}^{1,3}=-x_2\mathbf{e}_1+x_1\mathbf{e}_2$$

$$\mathbf{V}^{1,4}=x_2\mathbf{e}_1+x_1\mathbf{e}_2,\quad \mathbf{V}^{1,5}=x_1\mathbf{e}_1-\alpha x_3\mathbf{e}_3,\quad \mathbf{V}^{1,6}=x_2\mathbf{e}_2-\alpha x_3\mathbf{e}_3$$

$$\mathbf{V}^{2,1}=-2x_1x_3\mathbf{e}_1+(x_1^2+\alpha x_3^2)\mathbf{e}_3,\quad \mathbf{V}^{2,2}=-x_2x_3\mathbf{e}_1-x_1x_3\mathbf{e}_2+x_1x_2\mathbf{e}_3$$

$$\mathbf{V}^{2,3}=-2x_2x_3\mathbf{e}_2+(x_2^2+\alpha x_3^2)\mathbf{e}_3,\quad \mathbf{V}^{2,4}=[x_1^2-(2+\alpha)x_3^2]\mathbf{e}_1-2\alpha x_1x_3\mathbf{e}_3$$

$$\mathbf{V}^{2,5}=x_1x_2\mathbf{e}_1-2^{-1}(1+\alpha)x_3^2\mathbf{e}_2-\alpha x_2x_3\mathbf{e}_3,\quad \mathbf{V}^{2,6}=(x_2^2-x_3^2)\mathbf{e}_1$$

$$\mathbf{V}^{2,7}=(x_1^2-x_3^2)\mathbf{e}_2,\quad \mathbf{V}^{2,8}=-2^{-1}(1+\alpha)x_3^2\mathbf{e}_1+x_1x_2\mathbf{e}_2-\alpha x_1x_3\mathbf{e}_3$$

$$\mathbf{V}^{2,9}=[x_2^2-(2+\alpha)x_3^2]\mathbf{e}_2-2\alpha x_2x_3\mathbf{e}_3;\quad \alpha\equiv\nu(1-\nu)^{-1}$$

The coefficients on the right-hand side of (3.2) are equal to the constants $g_3(0)$, $-g_{1,2}$, $g_{1,1}$, $g_{2,1}$, $2^{-1}g_{2,2}$, $g_{2,3}$, respectively, multiplied by $\pi E(1-\nu^2)^{-1}$. Suppose L has the dimension of length. Then the dimensions of the quantities A_0, B_i and C_{ij} are L^{-1}, L^{-2} and L^{-3} respectively. If Ox_3 is the axis of symmetry of the body Ω, where the parts of its boundary on which boundary conditions (1.5) and (1.6) are specified are also axisymmetrical, then

$$g_{1,1}=\dots=g_{1,4},\quad g_{1,5}=g_{1,6};\quad g_{2,1}=g_{2,3},\quad g_{2,2}=0,\quad g_{2,4}=\dots=g_{2,9}$$

Green's vector function $\mathbf{G}$ gives a solution of the problem of the action of a concentrated force on the boundary Ω. We introduce also the solutions $\mathbf{G}^{(i)}$ and $\mathbf{G}^{(m,n)}$ of the problem of the loading of a body Ω at the point O by point couples and polycouples, which satisfy Eq. (1.1) conditions (2.1) and (1.5) and the following relations

$$\mathbf{G}^{(i)}(\mathbf{x})=\mathbf{S}^{(i)}(\mathbf{x})+O(1),\quad |\mathbf{x}|\to 0;\quad \mathbf{G}^{(m,n)}(\mathbf{x})=\mathbf{S}^{(m,n)}(\mathbf{x})+O(1),\quad |\mathbf{x}|\to 0$$

$$\mathbf{S}^{(1)}(\mathbf{x})=-\frac{\partial\mathbf{T}(\mathbf{x})}{\partial x_2},\quad \mathbf{S}^{(2)}(\mathbf{x})=\frac{\partial\mathbf{T}(\mathbf{x})}{\partial x_1},\quad \mathbf{S}^{(m,n)}(\mathbf{x})=\frac{\partial^m\mathbf{T}(\mathbf{x})}{\partial x_1^{m-n}\partial x_2^n}$$

Here $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are the solutions of the problem of the loading of an elastic half-space by a unit point couple (the superscript indicates the direction of the intensity vector). For $\mathbf{G}^{(i)}$ and $\mathbf{G}^{(m,n)}$ representations of the form (2.5) and expansions similar to (3.1) and (3.2) are correct.

We will use the method of matched asymptotic expansions (see [11] and [12, Chapter 3, Section 1]) and we will write the asymptotic form of the regular component $\mathbf{g}(\varepsilon\boldsymbol{\eta};\mathbf{x})$ of Green's vector function $\mathbf{G}(\varepsilon\boldsymbol{\eta};\mathbf{x})$ for small values of the parameter ε. In the neighbourhood of the origin of coordinates we will introduce "expanded" coordinates

$$\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3);\quad \xi_i=\varepsilon^{-1}x_i \tag{3.3}$$

By (2.3) we have

$$\mathbf{T}(x_1-\varepsilon\eta_1,x_2-\varepsilon\eta_2,x_3)=\varepsilon^{-1}\mathbf{T}(\xi_1-\eta_1,\xi_2-\eta_2,\xi_3)$$

Using the expansion

$$\mathbf{T}(\xi_1-\eta_1,\xi_2-\eta_2,\xi_3)=\mathbf{T}(\boldsymbol{\xi})-\sum_{i=1}^{2}\eta_i\frac{\partial\mathbf{T}(\boldsymbol{\xi})}{\partial\xi_i}+\sum_{m=2}^{\infty}\frac{(-1)^m}{m!}\sum_{n=0}^{m}C_m^n\eta_1^{m-n}\eta_2^n\frac{\partial^m\mathbf{T}(\boldsymbol{\xi})}{\partial\xi_1^{m-n}\partial\xi_2^n} \tag{3.4}$$

which holds for fairly large $|\boldsymbol{\xi}|$, we can establish the following asymptotic expansion

$$\mathbf{g}(\varepsilon\boldsymbol{\eta};\mathbf{x})=\mathbf{g}(\mathbf{x})+\varepsilon[\eta_2\mathbf{g}^{(1)}(\mathbf{x})-\eta_1\mathbf{g}^{(2)}(\mathbf{x})]+\varepsilon^2\sum_{n=0}^{2}2^{-1}C_2^n\eta_1^{2-n}\eta_2^n\mathbf{g}^{(2,n)}(\mathbf{x})+\dots \tag{3.5}$$

Here $\mathbf{g}^{(i)}$ and $\mathbf{g}^{(m,n)}$ are the regular components of the singular vector functions $\mathbf{G}^{(i)}$ and $\mathbf{G}^{(m,n)}$. The next terms on the right in (3.5) are found using expansion (3.4). Note that this result can also be easily obtained by the method of combined asymptotic expansions [13, 14].

Eliminating the parameter ε, we can rewrite (3.5) in the form

$$\mathbf{g}(\mathbf{y};\mathbf{x}) = \mathbf{g}(\mathbf{x}) - y_1\mathbf{g}^{(2)}(\mathbf{x}) + y_2\mathbf{g}^{(1)}(\mathbf{x}) + \sum_{n=0}^{2} 2^{-1}C_2^n y_1^{2-n} y_2^n \mathbf{g}^{(2,n)}(\mathbf{x}) + \dots \tag{3.6}$$

4. THE ASYMPTOTIC FORM OF THE CONTACT PRESSURE

We will introduce extended coordinates (3.3) into Eq. (2.6) and we will make the replacement of variables $y_i = \varepsilon\eta_i$ $(i = 1, 2)$ in the integrals. The parameter ε then vanishes from the equation of the boundary of the integration region

$$\frac{1-\nu^2}{\pi E}\iint_\omega \frac{\varepsilon p(\varepsilon\eta_1,\varepsilon\eta_2)d\eta_1 d\eta_2}{\sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2}} =$$
$$= \delta_0 + \varepsilon(\beta_1\xi_2 - \beta_2\xi_1) - \varepsilon\iint_\omega g_3(\varepsilon\boldsymbol{\eta};\varepsilon\xi_1,\varepsilon\xi_2,0)\varepsilon p(\varepsilon\boldsymbol{\eta})d\boldsymbol{\eta} \tag{4.1}$$

The following expansion is obtained from (4.1) for the contact pressure $p(x_1, x_2) \equiv -\sigma_{33}(\mathbf{u}; x_1, x_2, 0)$

$$p(x_1,x_2) = \varepsilon^{-1}p^0(\xi_1,\xi_2) + p^1(\xi_1,\xi_2) + \dots \tag{4.2}$$

Correspondingly, for the resultant and moments of the load system acting on the punch, we have

$$F_3 = \varepsilon F_3^0 + \varepsilon^2 F_3^1 + \dots;\quad M_i = \varepsilon^2 M_i^0 + \varepsilon^3 M_i^1 + \dots,\quad i = 1,2 \tag{4.3}$$

We replace the kernel of the integral operator n the right-hand side of (4.1) by expansion (3.5), and we then use expansions of the form (3.2) and take into account the replacement of variables (3.3). We substitute series (4.2) into the relation obtained, equate terms of like powers of the parameter ε and obtain the equations

$$(Bp^0)(\xi_1,\xi_2) \equiv \frac{1-\nu^2}{\pi E}\iint_\omega \frac{p^0(\eta_1,\eta_2)d\eta_1 d\eta_2}{\sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2}} = \delta_0 \tag{4.4}$$

$$(Bp^1)(\xi_1,\xi_2) = -\tilde{F}_3^0 A_0 + \beta_1\xi_2 - \beta_2\xi_1 \tag{4.5}$$

$$(Bp^2)(\xi_1,\xi_2) = -\tilde{F}_3^0(B_1\xi_1 + B_2\xi_2) - \sum_{i=1}^{2}\tilde{M}_i^0 A_0^{(i)} - \tilde{F}_3^1 A_0 \tag{4.6}$$

$$(Bp^3)(\xi_1,\xi_2) = -\tilde{F}_3^0(C_{11}\xi_1^2 + 2C_{12}\xi_1\xi_2 + C_{22}\xi_2^2) - \sum_{i=1}^{2}\tilde{M}_i^0(B_1^{(i)}\xi_1 + B_2^{(i)}\xi_2) -$$
$$-\tilde{F}_3^1(B_1\xi_1 + B_2\xi_2) - \sum_{n=0}^{2}\tilde{M}_{2,n}^0 A_0^{(2,n)} - \sum_{i=1}^{2}\tilde{M}_i^1 A_0^{(i)} - \tilde{F}_3^2 A_0 \tag{4.7}$$

The coefficients $\tilde{F}_3^r$, $\tilde{M}_i^r$ and $\tilde{M}_{2,n}^r$ are equal to the quantities $(\pi E)^{-1}(1-\nu^2)$ and F_3^r, M_i^r multiplied by $M_{2,n}^r$, calculated from the formulae

$$F_3^r = \iint_\omega p^r(\boldsymbol{\eta})d\boldsymbol{\eta},\quad \begin{Bmatrix} M_1^r \\ M_2^r \end{Bmatrix} = \iint_\omega \begin{Bmatrix} \eta_2 \\ -\eta_1 \end{Bmatrix} p^r(\boldsymbol{\eta})d\boldsymbol{\eta}$$

$$M_{m,n}^r = \frac{(-1)^m}{m!}C_m^n\iint_\omega \eta_1^{m-n}\eta_2^n p^r(\boldsymbol{\eta})d\boldsymbol{\eta},\quad m = 2,3,\dots$$

The constants $A_0^{(i)}$, $B_k^{(i)}$ $(i, k = 1, 2)$ and $A_0^{(2,n)}$ $(n = 0, 1, 2)$ are the coefficients in the asymptotic expansion (analogous to (3.2)) of the component of the regular component of the vectors $\mathbf{G}^{(i)}$ and $\mathbf{G}^{(2,n)}$ respectively, orthogonal to the boundary. The quantities A_0 and $B_k^{(i)}$, $A_0^{(2,n)}$ have dimensions of L^{-2} and L^{-3}. They depend, generally speaking, on Poisson's ratio and are determined by the shape and dimensions of the body Ω and its attachment conditions.

Hence, the problem of constructing the asymptotic form of the contact pressure reduces to recurrence series of equations (4.4)–(4.7), etc. for the terms of series (4.2). This result was obtained for the first time in [2] in a special case (see also [6, Section 53]).

5. THE PRESSURE UNDER AN ELLIPTIC PUNCH

Suppose the figure $\omega(\varepsilon)$ is bounded by an ellipse $x_1^2 + x_2^2(1-e^2)^{-1} = a_\varepsilon^2$, where e is the eccentricity and $a_\varepsilon = \varepsilon A$ is the semimajor axis. Then, using the solutions in [15], from (4.4)–(4.6) we obtain the trinomial asymptotic representation (4.2) in the form (reverting to the real scale)

$$p(x_1, x_2) \simeq \frac{E}{2(1-\nu^2)a\sqrt{1-e^2}}\left(1-\frac{x_1^2}{a^2}-\frac{x_2^2}{a^2(1-e^2)}\right)^{-1/2} \times$$
$$\times\left\{\frac{\delta_0}{\mathbf{K}(e)}-\frac{aA_0}{\mathbf{K}(e)}\delta_0-\frac{\beta_2 x_1}{\mathbf{D}(e)}+\frac{\beta_1 x_2}{\mathbf{B}(e)}+\left(\frac{aA_0}{\mathbf{K}(e)}\right)^2\delta_0-\frac{a\delta_0}{\mathbf{K}(e)}\left[\frac{B_1 x_1}{\mathbf{D}(e)}+\frac{B_2 x_2}{\mathbf{B}(e)}\right]\right\} \tag{5.1}$$
$$\mathbf{D}(e) = e^{-2}[\mathbf{K}(e)-\mathbf{E}(e)], \quad \mathbf{B}(e) = \mathbf{K}(e)-\mathbf{D}(e)$$

Here $\mathbf{K}$ and $\mathbf{E}$ are the complete elliptic integrals of the first and second kind, and the values of the coefficients on the right-hand side of Eq. (4.6) are

$$\tilde{F}_3^0 = A\mathbf{K}(e)^{-1}\delta_0, \quad \tilde{M}_1^0 = \tilde{M}_2^0 = 0$$
$$\tilde{M}_{2,0}^0 = A^3[6\mathbf{K}(e)]^{-1}\delta_0, \quad \tilde{M}_{2,1}^0 = 0, \quad \tilde{M}_{2,2}^0 = \tilde{M}_{2,0}^0(1-e^2)$$
$$\tilde{F}_3^1 = -A^2\mathbf{K}(e)^{-2}A_0\delta_0, \quad \tilde{M}_1^1 = A^3(1-e^2)[3\mathbf{B}(e)]^{-1}\beta_1 \tag{5.2}$$
$$\tilde{M}_2^1 = A^3[\mathbf{D}(e)]^{-1}\beta_2; \quad \tilde{F}_3^2 = A^3\mathbf{K}(e)^{-3}A_0^2\delta_0$$

This example shows that the calculation of the first terms of the asymptotic form is somewhat simplified if the origin of coordinates coincides with the so-called [16] centre of pressure of the punch base. In this case $M_1^0 = M_2^0 = 0$.

Remark. It is also easy to write the solution of Eq. (4.7) (see [2, 6, 8, 15, 17]). In this case the solution of the equation $(Bp)(\xi_1, \xi_2) = \xi_1\xi_2$ can be obtained from the well-known results [15, Chapter 5, Section 8] in the form

$$p(\xi_1, \xi_2) = \frac{E}{2(1-\nu^2)A\sqrt{1-e^2}}\left(1-\frac{\xi_1^2}{A^2}-\frac{\xi_2^2}{A^2(1-e^2)}\right)^{-1/2}\frac{\xi_1\xi_2}{\mathbf{D}(e)-\mathbf{C}(e)}$$
$$\mathbf{C}(e) = e^{-2}[\mathbf{D}(e)-\mathbf{B}(e)]$$

6. THE EQUATION OF THE CONTACT PROBLEM FOR A BODY OF FINITE DIMENSIONS

Using expansion (3.6), we obtain

$$\iint_{\omega(\varepsilon)} g_3(\mathbf{y}; x_1, x_2, 0)p(\mathbf{y})d\mathbf{y} = F_3 g_3(x_1, x_2, 0) +$$
$$+\sum_{i=1}^{2} M_i g_3^{(i)}(x_1, x_2, 0) + \sum_{n=0}^{2} M_{2,n} g_3^{(2,n)}(x_1, x_2, 0) + \ldots$$

We will convert the previous relation taking expansions of the form (3.2) into account, bearing in mind the distribution of the quantities F_3, M_i, $M_{2,n}\ldots$ in powers of the parameter ε (see (4.3)). We have

$$\iint\limits_{\omega(\varepsilon)} g_3(\mathbf{y}; x_1, x_2, 0)p(\mathbf{y})d\mathbf{y} = \tilde{F}_3 A_0 + \tilde{F}_3(B_1 x_1 + B_2 x_2) + \sum_{i=1}^{2} \tilde{M}_i A_0^{(i)} +$$
$$+\tilde{F}_3(C_{11}x_1^2 + 2C_{12}x_1 x_2 + C_{22}x_2^2) + \sum_{i=1}^{2} \tilde{M}_i(B_1^{(i)}x_1 + B_2^{(i)}x_2) + \sum_{n=0}^{2} \tilde{M}_{2,n} A_0^{(2,n)} + \ldots \tag{6.1}$$

It is clear that the substitution of expansion (4.1) into (6.1) leads to Eqs (4.4)–(4.7). On the other hand, retaining a finite number of terms in (6.1) we obtain a single "coupled" equation, the solution of which reduces to the solution of a system of linear algebraic equations. Note that the method of reducing the integral equation of contact problem (2.6) to an approximate equation by means of a polynomial approximation of the kernel of the integral operator on its right-hand side was proposed previously in [18] (see also [6, Section 54]).

Example. We will confine ourselves solely to the first term on the right-hand side of (6.1). The solution of the equation

$$(Bp)(x_1, x_2) = \delta_0 - F_3 A_0 + \beta_1 x_2 - \beta_2 x_1$$

differs from (5.1) in that the expression in braces is replaced by

$$\frac{\delta_0 - F_3 A_0}{\mathbf{K}(e)} - \frac{\beta_2 x_1}{\mathbf{D}(e)} + \frac{\beta_1 x_2}{\mathbf{B}(e)}$$

Substituting this solution into the equilibrium equation of the punch we obtain

$$F_3 = \frac{a\delta_0}{\mathbf{K}(e)}\left(1 + \frac{aA_0}{\mathbf{K}(e)}\right)^{-1}$$

which gives three correct terms of the asymptotic expansion of F_3 (see (5.2)). This effect was pointed out previously in [19].

This research was supported financially by the St Petersburg Administration (M98-2.2P-36).

REFERENCES

1. ALEKSANDROV, V. M., Asymptotic methods in contact problems of elasticity theory. *Prikl. Mat. Mekh.*, 1968, **32**, 4, 672–683.
2. ALEKSANDROV, V. M. and VOROVICH, I. I., The action of a punch on an elastic layer of finite thickness. *Prikl. Mat. Mekh.*, 1960, **24**, 2, 323–333.
3. LUBYAGIN, I. A., POZHARSKII, D. A. and CHEBAKOV, M. I., The indentation of a punch in the shape of an elliptic paraboloid into an elastic three-dimensional wedge. *Prikl. Mat. Mekh.*, 1992, **56**, 2, 286–295.
4. BORODACHEV, N. M., The indentation of a punch into the end of a semi-infinite elastic cylinder. *Prikl. Mat. Mekh.*, 1967, **3**, 9, 83–89.
5. NOWACKI, W., *The Theory of Elasticity*. Mir, Moscow, 1975.
6. VOROVICH, I. I., ALEKSANDROV, V. M. and BABESHKO, V. A., *Non-classical Mixed Problems of the Theory of Elasticity*. Nauka, Moscow, 1974.
7. KRAVCHUK, A. S., The solution of contact problems with known Green function. *Prikl. Mat. Mekh.*, 1982, **46**, 2, 283–288.
8. ALEKSANDROV, V. M. and POZHARSKII, D. A., Non-classical Three-Dimensional Problems of the Mechanics of Contact Interactions of Elastic Bodies. *Faktorial*, Moscow, 1998.
9. MIKHLIN, S. G., *Linear Partial Differential Equations*. Vysshaya Shkola, Moscow, 1977.
10. ARUTYUNYAN, N. Kh., MOVCHAN, A. B. and NAZAROV, S. A., The behaviour of the solutions of problems of the theory of elasticity in unbounded regions with paraboloidal and cylindrical inclusions or cavities. *Uspekhi Mekh.*, 1987, **10**, 4, 3–91.
11. VAN DYKE, M. D., *Perturbation Methods in Fluid Mechanics.* New York: Academic Press, 1964.
12. IL'IN, A. M., *Matched Asymptotic Expansions of the Solutions of Boundary-value Problems*. Nauka, Moscow, 1989.
13. MAZ'YA, V. G., NAZAROV, S. A. and PLAMENEVSKII, B. A., *The Asymptotic Form of the Solutions of Elliptic Boundary-value Problems for Singular Perturbation of the Region*. Izd. Tbilis. Univ., Tbilisi, 1981.
14. NAZAROV, S. A., *Introduction to Asymptotic Methods of Elasticity Theory*. Leningr. Gos. Univ., Leningrad, 1983.
15. LUR'YE, A. I., *Three-dimensional Problems of Elasticity Theory*. Gostekhizdat, Moscow, 1955.
16. ARGATOV, I. I. and NAZAROV, S. A., The asymptotic solution of the problem of an elastic body lying on several small supports. *Prikl. Mat. Mekh.*, 1994, **58**, 2, 110–118.
17. DOVNOROVICH, V. I., *Three-dimensional Contact Problems of Elasticity Theory*. Izd. Belorus. Univ., Minsk, 1959.
18. ALEKSANDROV, V. M., Some contact problems for an elastic layer. *Prikl. Mat. Mekh.*, 1963, **27**, 4, 758–764.
19. NAZAROV, S. A. and POLYAKOVA, O. R., Fracture criteria, asymptotic conditions at the crack tips and self-conjugate expansion of the Lamé operator. *Trudy Mosk. Mat. Obshch.*, 1996, **57**, 16–75.

Translated by R.C.G.